

# WEAK LEFSCHETZ PROPERTY AND STELLAR SUBDIVISIONS OF GORENSTEIN COMPLEXES

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**ABSTRACT.** Assume  $\sigma$  is a face of a Gorenstein\* simplicial complex  $D$ . We investigate the question of whether the Weak Lefschetz Property of the Stanley–Reisner ring  $k[D]$  (over an infinite field  $k$ ) is equivalent to the same property of the Stanley–Reisner ring  $k[D_\sigma]$  of the stellar subdivision  $D_\sigma$ . We prove that this is the case if the dimension of  $\sigma$  is big compared to the codimension.

## CONTENTS

1. Introduction	1
2. Notation	2
3. Some general lemmas	5
4. Weak Lefschetz Property and stellar subdivisions	8
4.1. Proof of Lemma 4.10	15
4.2. Proof of Lemma 4.15	16
4.3. Proof of Theorem 4.4	18
5. Further results on the general question	18
5.1. An interesting example not coming from simplicial complexes	24
5.2. Final Remarks	24
References	25

## 1. INTRODUCTION

An important open question in algebraic combinatorics is whether for a simplicial sphere, or more generally for a Gorenstein\* simplicial complex, the  $f$ -vector satisfies McMullen’s  $g$ -conjecture. For details see for example [16] or [6, Section 5.6]. It is well-known that for the  $g$ -conjecture to hold it is enough to prove that the Stanley–Reisner ring  $\mathbb{R}[D]$  of  $D$  over the real numbers satisfies the Weak Lefschetz Property (WLP for short). Actually Richard Stanley [18] proved that if  $D$  is the boundary complex of a convex simplicial polytope it holds that  $\mathbb{R}[D]$  satisfies the even stronger Strong Lefschetz Property (SLP for short).

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Eric Babson and Eran Nevo [3] proved that if  $k$  is an infinite field of characteristic 0,  $D$  is a homology sphere with  $k[D]$  SLP and  $\sigma$  is a face of  $D$  with  $k[L]$  SLP (where  $L$  denotes the link of  $\sigma$  in  $D$ ) then it follows that  $k[D_\sigma]$  has the SLP, where  $D_\sigma$  denotes the stellar subdivision of  $D$  with respect to  $\sigma$ . We investigate similar questions for  $D$  a Gorenstein\* simplicial complex and SLP replaced by WLP. Using constructions motivated by the interpretation of stellar subdivision in terms of Kustin–Miller unprojections [1], we prove in Corollary 4.5 that if  $2(\dim \sigma) > \dim D + 1$  and  $k$  denotes an infinite field then the Stanley–Reisner ring  $k[D]$  has the WLP if and only if  $k[D_\sigma]$  has the WLP. In addition, in Corollary 4.3 we prove that if  $k[L]$  has the SLP and  $k[D]$  has the WLP (or more generally if  $K[D]$  has the WLP and  $k[L]$  satisfies a certain property we call  $M_{q,p_1}$ , see Theorem 4.2) then it follows that  $k[D_\sigma]$  has the WLP.

Section 2 introduces the basic notations, while Section 3 presents a number of general lemmas we need. Section 4 is the core of the paper and provides statements and proofs of the main results. Section 5 contains some further results and constructions which could, perhaps, prove useful in attacking the problem of whether  $k[D]$  WLP is equivalent to  $k[D_\sigma]$  WLP without any assumptions on the dimension of  $\sigma$ . If this equivalence was to be proven, it would then have as corollary the  $g$ -conjecture for the class of PL-spheres, cf. [3, Remark 1.3.2].

We illustrate the structure of the arguments of the paper in Figure 1. The four central technical lemmas are shown in red, as well as Corollary 4.5. The results of Section 5, including an alternative proof of Corollary 4.5, are depicted in grey.

## 2. NOTATION

In the following  $k$  denotes an infinite field of arbitrary characteristic. All graded  $k$ -algebras will be commutative, Noetherian and of the form  $G = \bigoplus_{i \geq 0} G_i$  with  $G_0 = k$  and  $\dim_k G_i < \infty$  for all  $i$ . The *Hilbert function* of  $G$  is the function  $\mathrm{HF}(G) : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $m \mapsto \dim_k G_m$ . The  $k$ -algebra  $G$  is called standard graded if it is generated, as a  $k$ -algebra, by  $G_1$ . An element  $a \in G$  is called linear if  $a \in G_1$ . For a polynomial ring we use the notions of monomial order, initial term, initial ideal and reverse lexicographic order as defined in [5, Section 15]. If  $G$  is a standard graded  $k$ -algebra with  $\dim G = d$  and  $f_1, \dots, f_d \in G_1$  are (Zariski) general linear elements of  $G$  we call  $G/(f_1, \dots, f_d)$  a general Artinian reduction of  $G$ .

We say that an Artinian standard graded algebra  $F$  has the *Weak Lefschetz Property* (WLP for short) if for general  $\omega \in F_1$  and all  $i$  the multiplication by  $\omega$  map  $F_i \rightarrow F_{i+1}$  is of maximal rank, which means that it is injective or surjective (or both). It is well-known (see, for example, [3, Lemma 4.1]) that  $F$  has the WLP if and only if there exists  $a \in F_1$  such that for all  $i$  the multiplication by  $a$  map  $F_i \rightarrow F_{i+1}$  is of maximal rank.

We say that a standard graded  $k$ -algebra  $G$  with  $\dim G \geq 1$  has the WLP if it is Cohen–Macaulay and for general linear elements  $f_1, \dots, f_{\dim G}$  of  $G$  we have that the algebra  $G/(f_1, \dots, f_{\dim G})$ , which is Artinian by Lemma 3.3, has the WLP. Good general references for the Weak and Strong Lefschetz Properties are [8, 12]. By [8, Proposition 3.2], if  $F$  is an Artinian standard

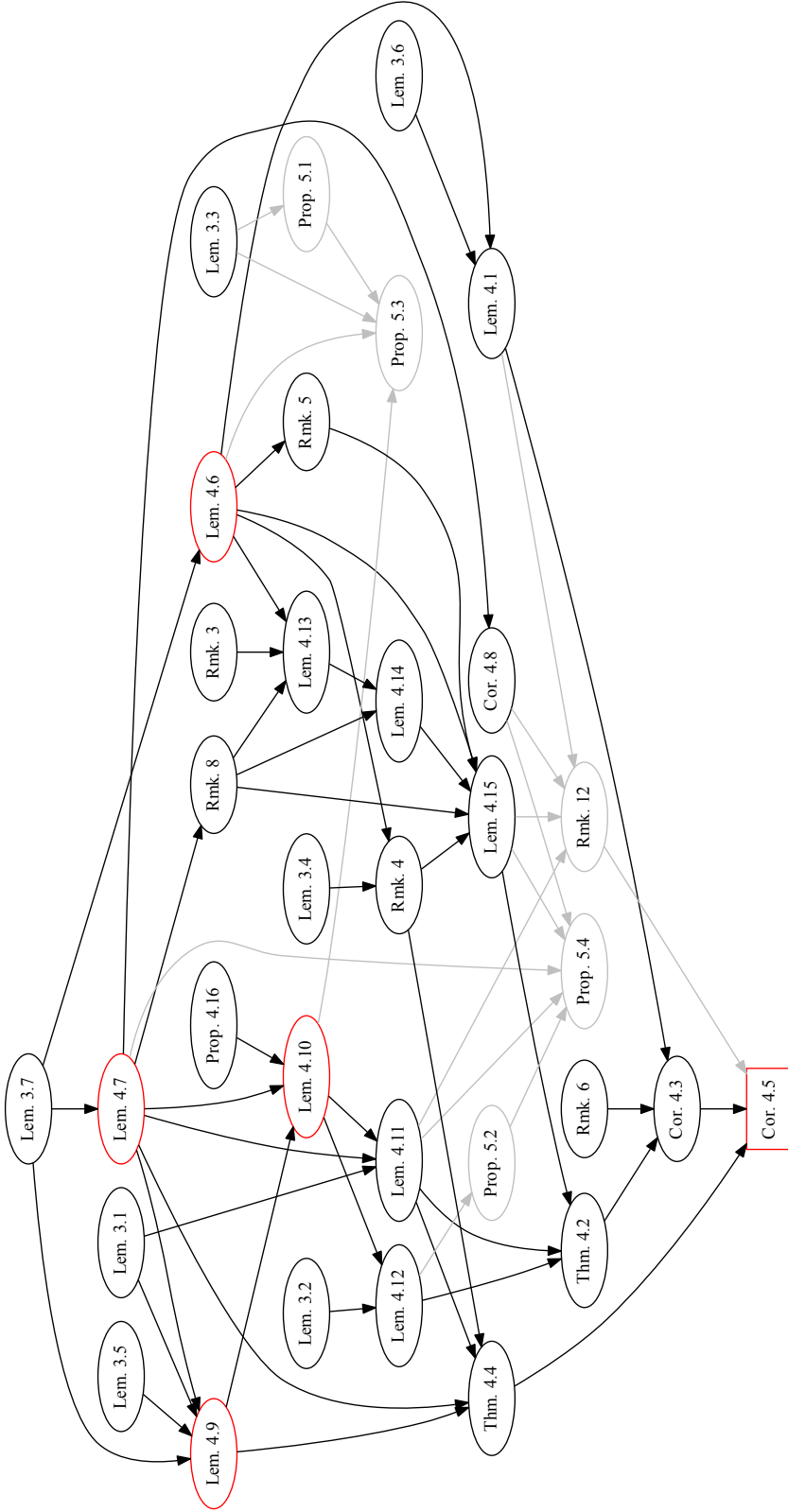


FIGURE 1. Structure of the arguments of the paper.

graded  $k$ -algebra with the WLP it follows that  $\text{HF}(F)$  is unimodal, which means that there is no triple  $j_1 < j_2 < j_3$  such that  $\text{HF}(j_1, F) > \text{HF}(j_2, F)$  and  $\text{HF}(j_3, F) > \text{HF}(j_2, F)$ .

We say that an Artinian standard graded algebra  $F = \bigoplus_{i=0}^r F_i$  with  $F_r \neq 0$  has the *Strong Lefschetz Property* (SLP for short) if  $\dim F_i = \dim F_{r-i}$  for all  $i$  with  $0 \leq i \leq r$  and for a general linear element  $\omega$  of  $F$  and all  $i$  with  $0 \leq 2i \leq r$ , the multiplication by  $\omega^{r-2i}$  map  $F_i \rightarrow F_{r-i}$  is bijective. We say that a standard graded  $k$ -algebra  $G$  with  $\dim G \geq 1$  has the SLP if it is Cohen–Macaulay and for general linear elements  $f_1, \dots, f_{\dim G}$  of  $G$  we have that the algebra  $G/(f_1, \dots, f_{\dim G})$ , which is Artinian by Lemma 3.3, has the SLP. If  $J \subset R$  is an ideal, we say that  $J$  has the WLP (resp. the SLP) if  $R/J$  has the WLP (resp. the SLP).

For a function  $h : \mathbb{Z} \rightarrow \mathbb{Z}$  we define

$$\Delta(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto h(m) - h(m-1).$$

For  $q > 0$  we inductively define  $\Delta^q(h) : \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\Delta^1(h) = \Delta(h)$  and  $\Delta^q(h) = \Delta^{q-1}(\Delta(h))$  for  $q > 1$ . Assume  $G$  is a standard graded  $k$ -algebra and  $a_1, a_2, \dots, a_p$  is a regular sequence in  $G$  consisting of linear elements, then  $\text{HF}(G/(a_1, \dots, a_p)) = \Delta^p(\text{HF}(G))$ .

We also define

$$\Delta^+(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto \max(0, h(m) - h(m-1)).$$

Assume  $F$  is an standard graded Artinian  $k$ -algebra. Then  $F$  has the WLP if and only if for general  $\omega \in F_1$  we have  $\text{HF}(F/(\omega)) = \Delta^+(\text{HF}(F))$ .

Assume that  $h : \mathbb{Z} \rightarrow \mathbb{Z}$  has the property that there exists  $m_0 \in \mathbb{Z}$  such that  $h(m) = 0$  when  $m < m_0$ . We define

$$\Gamma(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto \sum_{i=-\infty}^m h(i)$$

For  $q > 0$  we inductively define  $\Gamma^q(h) : \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\Gamma^1(h) = \Gamma(h)$  and  $\Gamma^q(h) = \Gamma^{q-1}(\Gamma(h))$ . Assume  $G$  is a standard graded  $k$ -algebra and  $T_1, \dots, T_p$  are new variables of degree 1, then  $\text{HF}(G[T_1, \dots, T_p]) = \Gamma^p(\text{HF}(G))$ .

For a graded  $k$ -algebra  $G$  we denote by  $\text{depth } G$  the depth of  $G$ . By [2, Theorem 1.2.8]

$$(1) \quad \text{depth } G = \min\{i : \text{Ext}_G^i(k, G) \neq 0\},$$

where  $k$  is considered as a  $G$ -module via  $k = G/(\bigoplus_{i \geq 1} G_i)$ . For an ideal  $I$  of a ring  $R$  and  $u \in R$  we write  $(I : u) = \{r \in R \mid ru \in I\}$  for the ideal quotient.

Assume  $A$  is a finite set. We set  $2^A$  to be the simplex with vertex set  $A$ , by definition it is the set of all subsets of  $A$ . A simplicial subcomplex  $D \subset 2^A$  is a subset with the property that if  $\tau \in D$  and  $\sigma \subset \tau$  then  $\sigma \in D$ . The elements of  $D$  are also called faces of  $D$ , and the dimension of a face  $\tau$  of  $D$  is one less than the cardinality of  $\tau$ . A facet of  $D$  is a maximal face of  $D$  with respect to (set-theoretic) inclusion. The dimension of  $D$  is the maximum dimension of a facet of  $D$ . We define the support of  $D$  by

$$\text{supp } D = \{i \in A \mid \{i\} \in D\}.$$

We denote by  $R_A$  the polynomial ring  $k[x_a \mid a \in A]$  with the degrees of all variables  $x_a$  equal to 1. For a simplicial subcomplex  $D \subset 2^A$  we define the *Stanley-Reisner ideal*  $I_{D,A} \subset R_A$  to be the ideal generated by the square-free monomials  $\prod_{i=1}^p x_{i_i}$  where  $\{i_1, i_2, \dots, i_p\}$  is not a face of  $D$ . In particular,  $I_{D,A}$  contains linear polynomials if and only if  $\text{supp } D \neq A$ . The *Stanley-Reisner ring*  $k[D, A]$  is defined by  $k[D, A] = R_A/I_{D,A}$ . For a nonempty face  $\sigma$  of  $D$  we set  $x_\sigma = \prod_{i \in \sigma} x_i \in R_A$ . For a nonempty finite set  $A$ , we set  $\partial A = 2^A \setminus \{A\} \subset 2^A$  to be the boundary complex of the simplex  $2^A$ . In the following, when the set  $A$  is clear we will simplify the notation  $k[D, A]$  to  $k[D]$ .

Assume that, for  $i = 1, 2$ ,  $D_i \subset 2^{A_i}$  is a subcomplex and the finite sets  $A_1, A_2$  are disjoint. By the join  $D_1 * D_2$  of  $D_1$  and  $D_2$  we mean the subcomplex  $D_1 * D_2 \subset 2^{A_1 \cup A_2}$  defined by

$$D_1 * D_2 = \{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in D_1, \alpha_2 \in D_2\}.$$

If  $\sigma$  is a face of  $D \subset 2^A$  we define the link of  $\sigma$  in  $D$  to be the subcomplex

$$\text{lk}_D \sigma = \{\alpha \in D \mid \alpha \cap \sigma = \emptyset \text{ and } \alpha \cup \sigma \in D\} \subset 2^{A \setminus \sigma}.$$

It is clear that the Stanley-Reisner ideal of  $\text{lk}_D \sigma$  is equal to the intersection of the ideal  $(I_{D,A} : x_\sigma)$  with the subring  $R_{A \setminus \sigma}$  of  $R_A$ . In other words, it is the ideal of  $R_{A \setminus \sigma}$  generated by the minimal monomial generating set of  $(I_{D,A} : x_\sigma)$ . Furthermore, we define the star of  $\sigma$  in  $D$  to be the subcomplex

$$\text{star}_D \sigma = \{\alpha \in D \mid \alpha \cup \sigma \in D\} \subset 2^A.$$

If  $\sigma$  is a nonempty face of  $D \subset 2^A$  and  $j \notin A$ , we define the *stellar subdivision*  $D_\sigma$  with new vertex  $j$  to be the subcomplex

$$D_\sigma = (D \setminus \text{star}_D \sigma) \cup (2^{\{j\}} * \text{lk}_D \sigma * \partial \sigma) \subset 2^{A \cup \{j\}}.$$

Following [16, p. 67], we say that a subcomplex  $D \subset 2^A$  is *Gorenstein\** over  $k$  if  $A = \text{supp } D$ ,  $k[D]$  is Gorenstein, and for every  $i \in A$  there exists  $\sigma \in D$  with  $\sigma \cup \{i\}$  not a face of  $D$ . The last condition combinatorially means that  $D$  is not a join of the form  $2^{\{i\}} * E$ , and algebraically that  $x_i$  divides at least one element of the minimal monomial generating set of  $I_{D,A}$ .

Assume  $D \subset 2^A$  is a Gorenstein\* simplicial complex and  $\sigma$  is a face of  $D$ . Set  $L = \text{lk}_D \sigma$ . It is well known (cf. [16, Section II.5]) that the subcomplex  $L \subset 2^{\text{supp } L}$  is Gorenstein\* with  $\dim L = \dim D - \dim \sigma - 1$ .

### 3. SOME GENERAL LEMMAS

In the present section we put together a number of general lemmas we need. Of particular interest is the following Lemma 3.1. It states that under certain conditions a non-general Artinian reduction of a WLP  $k$ -algebra inherits the WLP property, and plays a key role in the following.

**Lemma 3.1.** *Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  is a polynomial ring with all variables of degree 1 and  $T$  is a new variable of degree 1. Assume  $d \geq 1$ ,  $f_1, \dots, f_d \in R_1$  are  $d$  general linear elements, and  $J \subset R[T]$  is a homogeneous ideal with  $\dim R[T]/J = d$ . Assume  $R[T]/J$  has the WLP and is Cohen–Macaulay, and that  $\dim R[T]/(J + (f_1, \dots, f_d)) = 0$ . Then  $R[T]/(J + (f_1, \dots, f_d))$  has the WLP.*

*Proof.* Let  $f_1, \dots, f_{d+1} \in R_1$  be  $d+1$  general linear elements. For simplicity we set  $H = R[T]/(J + (f_1, \dots, f_d))$ . Since  $R[T]/J$  is Cohen–Macaulay of dimension  $d$  and  $\dim H = 0$  it follows that  $f_1, \dots, f_d$  is a regular sequence for  $R[T]/J$ . Hence

$$\mathrm{HF}(H) = \Delta^d(\mathrm{HF}(R[T]/J)).$$

Since (up to a nonzero constant)  $f_{d+1} + T$  is a general linear element of  $H$ ,  $H$  has the WLP if and only if  $\mathrm{HF}(H/(f_{d+1} + T)) = \Delta^+(\mathrm{HF}(H))$ . Since  $f_1, \dots, f_{d+1}$  are general linear elements of  $R$ , it follows that the ideal  $(f_1, \dots, f_d, f_{d+1} + T)$  of  $R[T]$  is equal to the ideal of  $R[T]$  generated by  $d+1$  general linear elements of  $R[T]$ . Hence, using the assumption that  $R[T]/J$  has the WLP we get

$$\begin{aligned} \mathrm{HF}(H/(f_{d+1} + T)) &= \mathrm{HF}(R[T]/(J + (f_1, \dots, f_d, f_{d+1} + T))) \\ &= \Delta^+(\Delta^d(\mathrm{HF}(R[T]/J))) \\ &= \Delta^+(\mathrm{HF}(H)). \end{aligned}$$

As a consequence,  $H$  has the WLP.  $\square$

**Remark 1.** The condition  $\dim R[T]/(J + (f_1, \dots, f_d)) = 0$  in the statement of Lemma 3.1 does not follow from the other assumptions since the  $f_i$  do not involve the variable  $T$ . For example, if  $R = k[x_1]$  and  $J = (Tx_1) \subset R[T]$  then the condition is not satisfied.

**Remark 2.** Is there a statement similar to Lemma 3.1 for the SLP ?

The following lemma is the analogue for the WLP of [14, Lemma 3.3] which is stated for the SLP and can be proven by the same arguments.

**Lemma 3.2.** (*Wiebe*) Assume  $R$  is a polynomial ring over an infinite field with all variables of degree 1,  $\tau$  is a monomial order on  $R$  and  $J \subset R$  is a homogeneous ideal with  $R/J$  Cohen–Macaulay. Denote by  $\mathrm{in}_\tau(J)$  the initial ideal of  $J$  with respect to  $\tau$ . We assume that  $R/\mathrm{in}_\tau(J)$  is Cohen–Macaulay and has the WLP. Then  $R/J$  has the WLP.

For a proof of the following lemma see [2, Proposition 1.5.12].

**Lemma 3.3.** Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  with all variables of degree 1,  $f_1, \dots, f_t \in R_1$  are  $t$  general linear elements of  $R$  and  $J \subset R$  is a homogeneous ideal. If  $t \leq \mathrm{depth} R/J$  then  $f_1, \dots, f_t$  is an  $R/J$ -regular sequence.

**Lemma 3.4.** Assume  $k$  is an infinite field and  $F = \bigoplus_{i=0}^d F_i$  is an Artinian standard graded Gorenstein  $k$ -algebra with  $F_d \neq 0$ . If  $d$  is even we set  $p_1 = d/2 - 1, p_2 = d/2$ , if  $d$  is odd we set  $p_1 = p_2 = (d-1)/2$ . Denote by  $\omega \in F_1$  a general linear element. Then the following are equivalent.

- i)  $F$  has the WLP.
- ii) The multiplication by  $\omega$  map  $F_{p_1} \rightarrow F_{p_1+1}$  is injective
- iii) The multiplication by  $\omega$  map  $F_{p_2} \rightarrow F_{p_2+1}$  is surjective.

*Proof.* It follows by [15, Remark 2.4]  $\square$

**Lemma 3.5.** Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  with all variables of degree 1. Assume  $J \subset R$  is a homogeneous ideal such that  $R/J$  is

Cohen–Macaulay and  $g_1, g_2 \in R$  are two nonzero linear elements. We define

$$\mathcal{S}_1 = \{c \in k : g_1 - cg_2 \notin J\} \subset \mathbb{A}^1,$$

$$\mathcal{S}_2 = \{c \in \mathcal{S}_1 : g_1 - cg_2 \text{ is } R/J\text{-regular}\} \subset \mathbb{A}^1$$

and

$$\mathcal{S}_3 = \{c \in \mathcal{S}_2 : R/(J + (g_1 - cg_2)) \text{ has the WLP}\} \subset \mathbb{A}^1.$$

Then, for all  $1 \leq i \leq 3$ , the subset  $\mathcal{S}_i \subset \mathbb{A}^1$  is Zariski open (but perhaps empty).

*Proof.* Denote by  $\mathcal{B}$  the finite dimensional vector space  $R_1$  considered as an affine variety. Consider the morphism  $\phi : \mathbb{A}^1 \rightarrow \mathcal{B}$ ,  $c \mapsto g_1 - cg_2$ . It is clear that the image of  $\phi$  is an affine subspace of  $\mathcal{B}$ , hence is Zariski closed. As a consequence, it is enough to prove that, for  $1 \leq i \leq 3$ , the three subsets

$$\mathcal{W}_1 = \{f \in \mathcal{B} : f \notin J\} \subset \mathcal{B},$$

$$\mathcal{W}_2 = \{f \in \mathcal{B} : f \text{ is } R/J\text{-regular}\}$$

and

$$\mathcal{W}_3 = \{f \in \mathcal{B} : f \text{ is } R/J\text{-regular and } R/(J + (f)) \text{ has the WLP}\} \subset \mathcal{B}$$

are Zariski open. For  $\mathcal{W}_1$  it is obvious. For  $\mathcal{W}_2$  it follows by the fact that the set of  $R/J$ -zero divisors is the union of the elements of the finite set consisting of the prime ideals associated to  $J \subset R$ . The case of  $\mathcal{W}_3$  is also well-known, see for example [3, Lemma 4.1].  $\square$

**Lemma 3.6.** *Assume  $e \geq 1$  is an integer and  $k$  is a field of characteristic 0 or of prime characteristic  $> e$ . Consider the polynomial ring  $R = k[x_1, \dots, x_n]$  with all variables of degree 1 and assume  $V \subset R$  is a  $k$ -vector subspace. If  $a^e \in V$  for all  $a \in R_1$  then it follows that  $R_e \subset V$ .*

*Proof.* According to [10, Section 3.2, Exercise 2], the linear span of the set  $\{a^e : a \in R_1\}$  is equal to  $R_e$ . The result follows.  $\square$

**Lemma 3.7.** *Assume  $D$  is a Gorenstein\* simplicial complex. Denote by  $k[D]$  the Stanley–Reisner ring of  $D$  over an infinite field  $k$  and by  $F$  a general Artinian reduction of  $k[D]$ . We have*

$$F = \bigoplus_{i=0}^{\dim k[D]} F_i$$

and  $F_{\dim k[D]}$  is 1-dimensional.

*Proof.* Since  $k$  is an infinite field and  $k[D]$  is Gorenstein, hence Cohen–Macaulay, by [2, proof of Theorem 5.1.10]  $\text{HF}(F)$  is equal to the  $h$ -vector of  $D$  (for a definition of the  $h$ -vector of a simplicial complex see [2, p. 205]). As a consequence, [2, Lemma 5.1.8] implies  $F_i = 0$  for  $i > \dim k[D]$ , and [2, Lemma 5.5.4] implies that  $F_{\dim k[D]} \neq 0$ , hence  $F_{\dim k[D]}$  is 1-dimensional by Gorenstein symmetry.  $\square$

## 4. WEAK LEFSCHETZ PROPERTY AND STELLAR SUBDIVISIONS

The present section contains our main results. In Corollary 4.5 we prove that if  $\sigma$  is a face of a Gorenstein\* simplicial complex  $D$  and  $2(\dim \sigma) > \dim D + 1$  then the Stanley–Reisner ring  $k[D]$  has the WLP if and only if  $k[D_\sigma]$  has the WLP. Moreover, in Corollary 4.3 we prove that if  $k[\text{lk}_D \sigma]$  has the SLP and  $k[D]$  has the WLP then it follows that  $k[D_\sigma]$  has the WLP.

We fix an infinite field  $k$  of arbitrary characteristic and a pair  $(D, \sigma)$ , where  $D$  is a Gorenstein\* simplicial complex with vertex set  $\{1, \dots, n\}$ , and  $\sigma = \{1, 2, \dots, q+1\}$  is a  $q$ -face of  $D$  with  $q \geq 1$ . We set  $d = \dim D + 1$ ,  $R = k[x_1, \dots, x_n]$  with the degrees of all variables equal to 1,  $x_\sigma = \prod_{i=1}^{q+1} x_i \in R$ . By  $I \subset R$  we denote the Stanley–Reisner ideal of  $D$ , hence  $k[D] = R/I$  and  $\dim R/I = d$ . We set  $I_L = (I : x_\sigma) \subset R$ , and  $f_1, \dots, f_{d+1}$  denote  $d+1$  general linear elements of  $R$ .

Moreover,  $T$  is a new variable of degree 1 and  $J_{st} = (I, x_\sigma, TI_L) \subset R[T]$  denotes the Stanley–Reisner ideal of the stellar subdivision  $D_\sigma$ , hence  $k[D_\sigma] = R[T]/J_{st}$ . We set

$$\begin{aligned} I_C &= (I, T^{q+1}, TI_L) \subset R[T], & I_G &= (I, T^{q+1} - x_\sigma, TI_L) \subset R[T], \\ A &= R/(I + (f_1, \dots, f_d)), & B &= R/(I_L + (f_1, \dots, f_d)), \\ C &= R[T]/(I_C + (f_1, \dots, f_d)), & G &= R[T]/(I_G + (f_1, \dots, f_d)). \end{aligned}$$

The rings  $R[T]/I_C$  and  $R[T]/I_G$  are closely related to the Kustin–Miller unprojection ring  $S$  appearing in [1, Theorem 1.1] and we view them as intermediate rings connecting  $k[D_\sigma]$  and  $k[D]$ . The rings  $A, B, C$  and  $G$  are Artinian reductions of  $k[D]$ ,  $R/I_L$ ,  $R[T]/I_C$  and  $R[T]/I_G$  respectively by linears that involve only the variables  $x_i$  but are, otherwise, general.

The basic properties of  $A, B$  and  $R/I_L$  are contained in Lemma 4.6, of  $C$  are contained in Lemma 4.7, of  $R[T]/I_G$  and  $G$  are contained in Lemma 4.9 and of  $R[T]/I_C$  are contained in Lemma 4.10. Since  $\dim k[D] = d$ , it follows that  $A$  is a general Artinian reduction of  $k[D]$ . Since by Lemma 4.6  $\dim R/I_L = d$ , it follows that  $B$  is a general Artinian reduction of  $R/I_L$ . On the other hand  $C$  is not a general Artinian reduction of  $R[T]/I_C$ , even though by Lemma 4.10  $\dim R[T]/I_C = d$ , since the  $f_i$  do not involve the variable  $T$ .

We denote by  $L \subset 2^{\{q+2, q+3, \dots, n\}}$  the link of  $\sigma$  in  $D$  and set  $k[L] = k[L, \{q+2, q+3, \dots, n\}]$ . Using that

$$(2) \quad R/I_L \cong (k[L])[x_1, \dots, x_{q+1}]$$

it follows that  $B$  is isomorphic to a general Artinian reduction of  $k[L]$ .

**Remark 3.** We will use the well-known fact, see for example [13, Proposition 3.3] or [8, Theorem 2.79], that if  $F = \bigoplus_{i=0}^r F_i$  with  $F_r \neq 0$  is a standard graded Gorenstein Artinian  $k$ -algebra then  $F_r$  is 1-dimensional, and for all  $i$  with  $0 \leq i \leq r$  the multiplication map  $F_i \times F_{r-i} \rightarrow F_r \cong k$  is a perfect pairing. We will refer to  $F_{r-i}$  as the Poincaré dual of  $F_i$ . As a consequence, given  $i, j$  with  $0 \leq i \leq j \leq r$  and  $0 \neq e \in F_i$  there exists  $e' \in F_{j-i}$  such that  $ee' \neq 0$  in  $F_j$ .

If  $d$  is even we set  $p_1 = d/2 - 1, p_2 = d/2$ , while if  $d$  is odd we set  $p_1 = p_2 = (d-1)/2$ .



**Remark 4.** Denote by  $\omega \in A_1$  a general linear element. Since by Lemma 4.6  $A$  is Gorenstein with  $A_i = 0$  for  $i \geq d + 1$  and  $A_d \neq 0$ , Lemma 3.4 implies that  $A$  has the WLP if and only if the multiplication by  $\omega$  map  $A_{p_1} \rightarrow A_{p_1+1}$  is injective and by the same lemma this happens if and only if the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective.

We say that  $k[L]$ , or equivalently  $R/I_L$  or equivalently  $B$ , has the property  $M_{q,p_1}$  if for a general element  $\omega \in B_1$  the multiplication by  $\omega^q$  map  $B_{p_1-q} \rightarrow B_{p_1}$  is injective.

**Remark 5.** Since by Lemma 4.6  $B$  is Artinian Gorenstein with  $B_i = 0$  for  $i \geq d - q$  and  $B_{d-q-1} \neq 0$ , by Gorenstein duality  $k[L]$  has the property  $M_{q,p_1}$  if and only if for a general element  $\omega \in B_1$  the multiplication map by  $\omega^q : B_{d-q-1-p_1} \rightarrow B_{d-q-1-(p_1-q)}$  is surjective. If  $d$  is even then  $d = 2p_1 + 2$ , hence  $d - q - 1 - p_1 = p_1 + 1 - q = p_2 - q$ , while if  $d$  is odd then  $d = 2p_1 + 1$ , hence  $d - q - 1 - p_1 = p_2 - q$ . Hence no matter if  $d$  is even or odd,  $k[L]$  has the property  $M_{q,p_1}$  if and only if for a general element  $\omega \in B_1$  the multiplication by  $\omega^q$  map  $B_{p_2-q} \rightarrow B_{p_2}$  is surjective.

**Remark 6.** Assume  $d$  is odd. Since  $d = 2p_1 + 1$ , we have  $d - q - 1 - (p_1 - q) = d - p_1 - 1 = p_1$ , hence the multiplication by  $\omega^q$  map in the definition of property  $M_{q,p_1}$  is between Poincaré dual graded components of  $B$ . Assume  $d$  is even. We have  $d = 2p_1 + 2$  and  $d - q - 1 - (p_1 - q) = d - p_1 - 1 = p_1 + 1$ , hence the multiplication by  $\omega^q$  map in the definition of the property  $M_{q,p_1}$  factors as  $f \circ g$ , where  $g$  is the multiplication by  $\omega$  map  $B_{p_1-q} \rightarrow B_{p_1-q+1}$  and  $f$  is multiplication by  $\omega^{q-1}$  map between the Poincaré duals  $B_{p_1-q+1}$  and  $B_{p_1}$ . As a consequence, no matter if  $d$  is even or odd if  $B$  has the SLP then it follows that property  $M_{q,p_1}$  holds for  $B$ .

**Lemma 4.1.** *If  $(q > p_1)$  or  $(q = p_1$  and the field  $k$  has characteristic 0 or a prime number  $> d - q - 1)$  then property  $M_{q,p_1}$  holds for  $k[L]$ .*

*Proof.* If  $q > p_1$  then  $B_{p_1-q} = 0$  and the result is obvious.

Assume  $q = p_1$ . Since  $B_0 = k$ , property  $M_{q,p_1}$  for  $k[L]$  is equivalent to  $\omega^{p_1} \neq 0$  in  $B$  for general  $\omega \in B_1$ . To get a contradiction we assume this property is not true, then it follows that  $b^{p_1} = 0$  for all  $b \in B_1$ . This, together with the assumptions on the characteristic of the field  $k$  imply by the general Lemma 3.6 that  $B_{p_1} = 0$ . Since  $d - q - 1 = d - p_1 - 1 \geq p_1$ , and  $B$  is standard graded we get  $B_{d-q-1} = 0$ . By Lemma 4.6  $B_{d-q-1} \neq 0$  which is a contradiction.  $\square$

**Theorem 4.2.** *Assume  $k[D]$  has the WLP and property  $M_{q,p_1}$  holds for  $k[L]$ . Then  $k[D_\sigma]$  has the WLP.*

*Proof.* By Lemma 4.11  $A$  has the WLP. Hence by Lemma 4.15  $C$  has the WLP. As a consequence, by Lemma 4.11  $R[T]/I_C$  has the WLP. Using Lemma 4.12 the result follows.  $\square$

**Corollary 4.3.** *i) Assume  $k[L]$  has the SLP. Then  $k[D]$  WLP implies that  $k[D_\sigma]$  has the WLP.*

*ii) Assume  $q > p_1$  or  $(q = p_1$  and the field  $k$  has characteristic 0 or a prime number  $> d - q - 1)$ . Then  $k[D]$  WLP implies that  $k[D_\sigma]$  has the WLP.*

*Proof.* Part i) follows from Theorem 4.2, since by Remark 6  $k[L]$  SLP implies that the property  $M_{q,p_1}$  holds for  $k[L]$ .

We prove Part ii). Using Lemma 4.1 property  $M_{q,p_1}$  holds for  $k[L]$ . Hence  $k[D_\sigma]$  has the WLP by Theorem 4.2.  $\square$

The proof of the following theorem will be given in Subsection 4.3.

**Theorem 4.4.** *Assume  $q > p_2$  and  $k[D_\sigma]$  has the WLP. Then  $k[D]$  has the WLP.*

**Corollary 4.5.** *Assume  $2(\dim \sigma) > \dim D + 1$ . Then  $k[D]$  has the WLP if and only if  $k[D_\sigma]$  has the WLP.*

*Proof.* We first prove that the statement  $2(\dim \sigma) > \dim D + 1$  is equivalent to  $q > p_2$ . Indeed, by the definitions  $q = \dim \sigma$ ,  $d = \dim D + 1$ . Assume  $d$  is even. Then  $p_2 = d/2$ . Hence  $q > p_2$  is equivalent to  $\dim \sigma > d/2$  which is equivalent to  $2(\dim \sigma) > d = \dim D + 1$ . Assume now  $d$  is odd. Then  $p_2 = (d-1)/2$ , hence  $q > p_2$  is equivalent to  $\dim \sigma > (d-1)/2$  which is equivalent to  $2(\dim \sigma) > d-1$ . But  $d$  odd implies  $d-1$  even, hence since  $2(\dim \sigma)$  is always even  $2(\dim \sigma) > d-1$  is equivalent to  $2(\dim \sigma) > d = \dim D + 1$ .

Assume  $2(\dim \sigma) > \dim D + 1$  and  $k[D]$  has the WLP. As we said above  $q > p_2$ . Since  $p_2 \geq p_1$ , we have  $q > p_1$ , hence Part ii) of Corollary 4.3 implies that  $k[D_\sigma]$  has the WLP.

Assume now  $2(\dim \sigma) > \dim D + 1$  and  $k[D_\sigma]$  has the WLP. As we said above  $q > p_2$ . By Theorem 4.4  $k[D]$  has the WLP.  $\square$

**Remark 7.** For a second proof of Corollary 4.5 see Remark 12.

**Lemma 4.6.** *The rings  $k[D]$ ,  $k[D_\sigma]$ ,  $A$  and  $B$  are Gorenstein.  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional.  $R/I_L$  is Gorenstein with  $\dim R/I_L = d$ . We have  $B = \bigoplus_{i=0}^{d-q-1} B_i$  and  $B_{d-q-1}$  is 1-dimensional. Moreover, for all  $m \geq 0$  we have*

$$\text{HF}(m, k[D_\sigma]) = \text{HF}(m, k[D]) + \sum_{i=1}^q \text{HF}(m-i, R/I_L).$$

*Proof.* Since by assumption  $D$  is Gorenstein\*, it follows that  $k[D]$  is Gorenstein. By definition  $A$  is a general Artinian reduction of  $k[D]$ , hence it is also Gorenstein. Since  $\dim k[D] = d$ , using Lemma 3.7 we get  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional.

By [17, p. 188] Gorenstein\* is a topological property. Hence a stellar subdivision of a Gorenstein\* simplicial complex is again Gorenstein\*. As a consequence  $k[D_\sigma]$  is Gorenstein.

As already mentioned in Section 2, we have that  $L \subset 2^{\text{supp } L}$  is Gorenstein\*, with  $\dim k[L] = d - q - 1$ . Using Equation (2) it follows that  $R/I_L$  is Gorenstein of dimension  $d$ . Since  $B$  is isomorphic to a general Artinian reduction of  $k[L]$ , it follows that  $B$  is Gorenstein. Moreover, Lemma 3.7 implies that  $B = \bigoplus_{i=0}^{d-q-1} B_i$  and  $B_{d-q-1}$  is 1-dimensional.

The equation between the Hilbert functions follows from [1, Remark 5].  $\square$

**Lemma 4.7.** (Recall  $\sigma$  is a  $q$ -face, with  $q \geq 1$ ) There is a well-defined bijective  $k$ -linear map of vector spaces

$$A \oplus B^q \rightarrow C, \quad ([a], [b_1], \dots, [b_q]) \mapsto [a + \sum_{i=1}^q b_i T^i]$$

for  $a, b_i \in R$ . As a corollary,  $C$  is Artinian and for all  $m \geq 0$

$$\text{HF}(m, C) = \text{HF}(m, A) + \sum_{i=1}^q \text{HF}(m - i, B).$$

Hence,  $\text{HF}(C)$  is equal to the Hilbert function of a general Artinian reduction of  $k[D_\sigma]$ . In particular  $C_i = 0$  for  $i \geq d + 1$  and  $C_d$  is 1-dimensional.

*Proof.* By definition  $C = R[T]/(I, TI_L, T^{q+1}, f_1, \dots, f_d)$ . We also have  $A = R/(I, f_1, \dots, f_d)$  and  $B = R/(I_L, f_1, \dots, f_d)$ . Denote by  $\phi$  the map in the statement of the present proposition.

$\phi$  is well-defined: Assume  $a, a', b_i, b'_i \in R$  with  $[a] = [a']$  in  $A$  and  $[b_i] = [b'_i]$  in  $B$  for all  $i$ . Then  $a - a' \in (I, f_1, \dots, f_d)$  and  $b_i - b'_i \in (I_L, f_1, \dots, f_d)$ , hence  $T(b_i - b'_i) \in (TI_L, f_1, \dots, f_d)$ . As a consequence

$$[a + \sum_{i=1}^q b_i T^i] = [a' + \sum_{i=1}^q b'_i T^i]$$

in  $C$ .

$\phi$  is surjective: Obvious from the definitions of  $\phi$  and  $C$ .

$\phi$  is injective: Assume  $a, b_i \in R$  with  $[a + \sum_{i=1}^q b_i T^i] = 0$  in  $C$ . This implies that there exist  $e_{a,1}, \dots, e_{a,r_1} \in I$ ,  $e_{b,1}, \dots, e_{b,r_2} \in I_L$ ,  $g_{a,1}, \dots, g_{a,r_1} \in R[T]$ ,  $g_{b,1}, \dots, g_{b,r_2} \in R[T]$ ,  $g_c \in R[T]$ ,  $g_{e,1}, \dots, g_{e,d} \in R[T]$  such that

$$a + \sum_{i=1}^q b_i T^i = \sum_{j=1}^{r_1} g_{a,j} e_{a,j} + T \sum_{j=1}^{r_2} g_{b,j} e_{b,j} + g_c T^{q+1} + \sum_{j=1}^d g_{e,j} f_j$$

with equality in  $R[T]$ . Looking at the coefficients of the powers of  $T$  we get  $a \in (I, f_1, \dots, f_d)$  and  $b_i \in (I_L, f_1, \dots, f_d)$  for all  $1 \leq i \leq q$ . Hence  $\phi$  is injective.

Since  $A, B$  are Artinian, they are finite dimensional  $k$ -vector spaces. Since  $\phi$  is surjective  $C$  is finite dimensional as a  $k$ -vector space which implies that  $C$  is Artinian.

The formula connecting the Hilbert functions of  $A, B, C$  is an immediate consequence of the fact that  $\phi$  is bijective.

Using [1, Remark 5] it follows that  $\text{HF}(C)$  is equal to the Hilbert function of a general Artinian reduction of  $k[D_\sigma]$ . As a consequence, the statements  $C_i = 0$  for  $i \geq d + 1$  and  $C_d$  is 1-dimensional follow from Lemma 3.7 applied to the Gorenstein\* simplicial complex  $D_\sigma$ .  $\square$

**Remark 8.** Taking graded components, Lemma 4.7 immediately implies that, for  $i \geq 0$ , there exists a  $k$ -vector space decomposition

$$C_i = A_i \oplus (\oplus_{j=1}^q B_{i-j} T^j).$$

Hence, if  $c \in C_i$  there exist unique  $a \in A_i$  and  $b_j \in B_j$  such that

$$c = a + b_{i-1} T + b_{i-2} T^2 + \dots + b_{i-q} T^q.$$

**Corollary 4.8.** *Assume that the Hilbert function of a general linear section of  $C$  is equal to the Hilbert function of a general linear section of a general Artinian reduction of  $k[D_\sigma]$ . Then  $C$  has the WLP if and only if  $k[D_\sigma]$  has the WLP.*

*Proof.* By Lemma 4.7  $\text{HF}(C)$  is equal to the Hilbert function of a general Artinian reduction of  $k[D_\sigma]$ . The result follows from the definition of WLP.  $\square$

Recall  $I_G = (I, T^{q+1} - x_\sigma, TI_L)$  and  $G = R[T]/(I_G + (f_1, \dots, f_d))$ .

**Lemma 4.9.** *i) The  $k$ -algebra  $R[T]/I_G$  is Gorenstein,  $\dim R[T]/I_G = d$  and  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$ .*

*ii) The ring  $G$  is Artinian Gorenstein, and  $\text{HF}(G) = \text{HF}(C)$ , which by Lemma 4.7 is equal to the HF of a general Artinian reduction of  $k[D_\sigma]$ .*

*iii) Assume  $k[D_\sigma]$  has the WLP. Then both  $R[T]/I_G$  and  $G$  have the WLP.*

*Proof.* Let  $z, z'$  be two new variables and  $c \in k$ . We set  $I_V = (I, TI_L, Tz - x_\sigma) \subset R[T, z]$  where  $\deg T = 1$ ,  $\deg z = q$ . We set  $\mathcal{M} = R[T, z, z']/(I_V)$ , where  $(I_V)$  is the ideal of  $R[T, z, z']$  generated by  $I_V$  and  $\deg T = \deg z' = 1$ ,  $\deg z = q$ . We also set  $\mathcal{Q} = \mathcal{M}/(z - T^{q-1}z')$  and

$$H_c = \mathcal{Q}/(z' - c^{q+1}T) = R[T]/(I, TI_L, c^{q+1}T^{q+1} - x_\sigma).$$

By [1, Theorem 1.1]  $R[T, z]/I_V$  is Gorenstein and  $\dim R[T, z]/I_V = d + 1$ . It follows that  $\mathcal{M}$  is Gorenstein and  $\dim \mathcal{M} = d + 2$ . Hence  $\dim \mathcal{Q} \geq d + 1$ . Since  $\mathcal{Q}/(z') = k[D_\sigma]$  which has dimension  $d$ , it follows that  $\dim \mathcal{Q} \leq d + 1$ . Hence  $\dim \mathcal{Q} = d + 1$ . Using that  $\mathcal{M}$  is Gorenstein, hence Cohen–Macaulay, it follows that  $z - T^{q-1}z'$  is an  $\mathcal{M}$ -regular element, hence  $\mathcal{Q}$  is Gorenstein.

Clearly

$$\mathcal{Q} = R[T, z']/(I, TI_L, T^q z' - x_\sigma),$$

hence  $\mathcal{Q}$  is standard graded. We have  $H_0 = k[D_\sigma]$ , hence  $\dim \mathcal{Q}/(z') = \dim \mathcal{Q} - 1$ . Since  $\mathcal{Q}$  is Gorenstein, hence Cohen–Macaulay, it follows that  $z'$  is a  $\mathcal{Q}$ -regular element.

Hence Lemma 3.5 implies that for general  $c \in k$  we have that  $z' - c^{q+1}T$  is a  $\mathcal{Q}$ -regular element, since the property is true for  $c = 0$ . As a consequence, for general  $c \in k$  the ring  $H_c$  is Gorenstein of dimension  $d$  and  $\text{HF}(H_c) = \text{HF}(k[D_\sigma])$ . For nonzero  $c$  using the linear change of coordinates  $T \mapsto cT$  we can assume that  $c = 1$ . Since  $R[T]/I_G$  is isomorphic to  $H_1$ , it follows that  $R[T]/I_G$  is Gorenstein,  $\dim R[T]/I_G = d$  and  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$ .

We now prove ii) We first prove that  $G$  is Artinian. The arguments used in the proof of Lemma 4.7 also give that there exists a well-defined surjective  $k$ -linear map of vector spaces

$$\psi : A \oplus B^q \rightarrow G, \quad ([a], [b_1], \dots, [b_q]) \mapsto [a + \sum_{i=1}^q b_i T^i]$$

for  $a, b_i \in R$ . Since  $A, B$  have finite dimension as  $k$ -vector spaces it follows that  $G$  has finite dimension as a  $k$ -vector space, hence it is Artinian. Since we proved that  $R[T]/I_G$  is Gorenstein, hence Cohen–Macaulay, and of dimension  $d$ ,  $G$  Artinian implies that  $f_1, \dots, f_d$  is a regular sequence for  $R[T]/I_G$ .

As a consequence, using that we proved above that  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$  it follows that

$$\text{HF}(G) = \Delta^d(\text{HF}(R[T]/I_G)) = \Delta^d(\text{HF}(k[D_\sigma])) = \text{HF}(C)$$

with the last equality by Lemma 4.7. Hence  $\dim_k G = \dim_k C$ . Using that by Lemma 4.7  $\dim_k C = (\dim_k A) + q(\dim_k B)$  we get that  $\dim_k G = (\dim_k A) + q(\dim_k B)$ . Since  $\psi$  is surjective it follows that  $\psi$  is bijective.

We now prove iii). Assume  $k[D_\sigma]$  has the WLP. By Lemma 3.5 we have that for general  $c \in k$  the algebra  $H_c$  has the WLP, since the property is true for  $c = 0$ . For nonzero  $c$  using the linear change of coordinates  $T \mapsto cT$  we can assume that  $c = 1$ . Hence  $R[T]/I_G$  has the WLP. Since by ii)  $G$  is Artinian, Lemma 3.1 implies that  $G$  has the WLP.  $\square$

Due to its length the proof of the following lemma will be given in Subsection 4.1.

**Lemma 4.10.** *The ring  $R[T]/I_C$  is Cohen–Macaulay with  $\dim R[T]/I_C = d$  and  $\text{HF}(R[T]/I_C) = \text{HF}(k[D_\sigma])$ . Moreover,  $I_C$  is the initial ideal of  $I_G$  with respect to the reverse lexicographic order in  $R[T]$  with  $T > x_1 > \dots > x_n$ .*

**Lemma 4.11.**  *$k[D]$  (resp.  $k[L]$ , resp.  $R[T]/I_C$ ) has the WLP if and only if  $A$  (resp.  $B$ , resp.  $C$ ) has the WLP.*

*Proof.* Since  $A$  is a general Artinian reduction of the Gorenstein  $k[D]$  it is immediate that  $A$  has the WLP if and only if  $k[D]$  has the WLP. Since  $B$  is a general Artinian reduction of the Gorenstein  $k[L]$  it is immediate that  $B$  has the WLP if and only if  $k[L]$  has the WLP.

By Lemma 4.10  $R[T]/I_C$  is Cohen–Macaulay of dimension  $d$ . Since by Lemma 4.7  $C$  is Artinian, it follows that  $f_1, \dots, f_d$  is an  $R[T]/I_C$ -regular sequence. Hence  $C$  WLP implies  $R[T]/I_C$  WLP. Conversely, assume that  $R[T]/I_C$  has the WLP. The result that  $C$  has the WLP follows by Lemma 3.1.  $\square$

**Lemma 4.12.** *Assume  $R[T]/I_C$  has the WLP. Then  $k[D_\sigma]$  has the WLP.*

*Proof.* We argue in a very similar way to [14, Proposition 2.2]. Assume  $R[T]/I_C$  has the WLP. The ordering we use in the polynomial ring  $R[T]$  is the reverse lexicographic ordering with  $T > x_1 > x_2 > \dots > x_n$ .

Consider the  $k$ -algebra automorphism  $\phi$  of  $R[T]$  defined by  $T \mapsto T, x_i \mapsto x_i + T$  for  $1 \leq i \leq q+1$  and  $x_i \mapsto x_i$  for  $q+2 \leq i \leq n$ . We claim that  $I_C$  is the initial ideal of  $\phi(J_{st})$ . Indeed, it is clear that  $I_C$  is a subset of the initial ideal of  $\phi(J_{st})$  and by Lemma 4.10  $\text{HF}(R[T]/I_C) = \text{HF}(k[D_\sigma])$ . It follows by Lemma 3.2 that  $R[T]/\phi(J_{st})$  has the WLP, hence also  $R[T]/J_{st} = k[D_\sigma]$  has the WLP.  $\square$

The following lemma will be used in the proof of Lemma 4.15. Since  $C$  is not Gorenstein, it does not follow from Remark 3.

**Lemma 4.13.** *Assume  $1 \leq i < p_1$  and  $0 \neq c \in C_i$ . Then there exists  $c' \in C_{p_1-i}$  such that  $cc' \neq 0$  in  $C_{p_1}$ . As a corollary, assume  $\omega \in C_1$  is any element, not necessarily general. If the multiplication by  $\omega$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective, it follows that for all  $i$  with  $1 \leq i \leq p_1$  the multiplication by  $\omega$  map  $C_i \rightarrow C_{i+1}$  is injective.*

*Proof.* Using Remark 8 write

$$c = a_i + \sum_{j=1}^q b_{i-j} T^j$$

(equality in  $C$ ) with  $a_i \in A_i$  and  $b_{i-j} \in B_{i-j}$  for all  $1 \leq j \leq q$ . Since  $c \neq 0$ , we have  $a_i \neq 0$  or  $b_{i-e} \neq 0$  for some  $e$  with  $1 \leq e \leq \min\{i, q\}$ .

Assume first that  $a_i \neq 0$ . By Lemma 4.6  $A$  is Artinian Gorenstein with  $A_d \neq 0$ . Hence by Remark 3 there exists  $a \in A_{p_1-i}$  such that  $a_i a \neq 0 \in A_{p_1}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = a$ .

For the rest of the argument we assume that there exists  $e$  with  $1 \leq e \leq \min\{i, q\}$  such that  $b_{i-e} \neq 0$  in  $B$ . We have two cases:

First Case: We assume  $p_1 \leq d - q - 1$ . By Lemma 4.6  $B$  is Artinian Gorenstein with  $B_{d-q-1} \neq 0$ . By Remark 3 there exists  $b \in B_{p_1-i}$  such that  $b_{i-e} b \neq 0$  in  $B_{p_1-e}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = b$ .

Second Case: We assume  $d - q - 1 < p_1$ . Hence  $d - p_1 - 1 < q$ . If  $d$  is even, since  $d = 2p_1 + 2$  we get  $p_1 + 1 < q$ . If  $d$  is odd, since  $d = 2p_1 + 1$  we get  $p_1 < q$ . Hence no matter if  $d$  is even or odd we have  $p_1 < q$ . Since  $e \leq i$  and  $i < p_1$  we get  $0 < p_1 - i < q$  and  $0 < p_1 + e - i < q$ . As a consequence  $0 \neq b_{i-e} T^e T^{p_1-i}$  in  $C_{p_1}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = T^{p_1-i}$ .

We now prove the corollary. We assume  $1 \leq i < p_1$ , that the multiplication by  $\omega$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and that the multiplication by  $\omega$  map  $C_i \rightarrow C_{i+1}$  is not injective and we will get a contradiction. By the assumptions there exists  $0 \neq c \in C_i$  such that  $\omega c = 0$  in  $C_{i+1}$ . By the first part of the present lemma there exists  $c' \in C_{p_1-i}$  such that  $cc' \neq 0$  in  $C_{p_1}$ . Hence by the assumptions  $\omega cc' \neq 0$  in  $C_{p_1+1}$ , which contradicts  $\omega c = 0$  in  $C_{i+1}$ .  $\square$

The ring  $C$  is not Gorenstein, hence we can not use Lemma 3.4. The following lemma is a substitute.

**Lemma 4.14.** *The following are equivalent*

- i)  $C$  has the WLP.
- ii) For general  $\omega \in R_1$  the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective.

*Proof.* Assume that i) holds. Since for nonzero  $c \in k$  the map  $C \rightarrow C$ , with  $x_i \mapsto x_i$  and  $T \mapsto cT$  is well-defined and an automorphism, it follows that for general  $\omega \in R_1$  we have that  $\omega + T$  is a general element of  $C_1$ . Since  $C$  is assumed to have the WLP to prove ii) it is enough to prove  $\dim C_{p_1} \leq \dim C_{p_1+1}$  and  $\dim C_{p_2+1} \leq \dim C_{p_2}$ . We assume it is not the case and we will get a contradiction.

By [8, Proposition 3.2]  $\text{HF}(C)$  is unimodal. Using Remark 8 it follows that  $\dim C_i = \dim C_{d-i}$  for all  $i \in \mathbb{Z}$ . First we assume that  $\dim C_{p_1} > \dim C_{p_1+1}$ . If  $d$  is odd then  $\dim C_{p_1} = \dim C_{p_1+1}$  which is a contradiction. Hence  $d$  is even. Since  $d - p_1 = p_1 + 2$  it follows that  $\dim C_{p_1+2} = \dim C_{p_1} > \dim C_{p_1+1}$ , which contradicts the unimodality of  $\text{HF}(C)$ .

We now assume that  $\dim C_{p_2+1} > \dim C_{p_2}$ . If  $d$  is odd then  $p_1 = p_2$ , and hence  $\dim C_{p_2} = \dim C_{p_2+1}$ , which is a contradiction. If  $d$  is even  $p_2 = p_1 + 1$  and  $d - (p_2 + 1) = p_1$ . Hence  $\dim C_{p_1} > \dim C_{p_1+1}$  which we proved above that can not happen.

Conversely assume that ii) holds. Then by Lemma 4.13 for all  $0 \leq i \leq p_1$  the multiplication by  $\omega + T$  map  $C_i \rightarrow C_{i+1}$  is injective. Since  $C$  is standard graded, the assumption that the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective implies that for all  $j \geq p_2$  the multiplication by  $\omega + T$  map  $C_j \rightarrow C_{j+1}$  is surjective. Since  $p_2 = p_1$  if  $d$  is odd and  $p_2 = p_1 + 1$  if  $d$  is even, we get that for all  $j \in \mathbb{Z}$  multiplication by  $\omega + T$  as a map  $C_j \rightarrow C_{j+1}$  is injective or surjective (or both), hence  $C$  has the WLP.  $\square$

Due to its length the proof of the following lemma will be given in Subsection 4.2.

**Lemma 4.15.** *The following are equivalent:*

- i)  $C$  has the WLP.
- ii)  $A$  has the WLP and property  $M_{q,p_1}$  holds for  $B$ .

**4.1. Proof of Lemma 4.10.** For the proof of Lemma 4.10 we need the following proposition.

**Proposition 4.16.** *Set  $\mathcal{H} = k[x_1, \dots, x_n, T, z]/(I, Tz, TI_L)$ . Then  $\mathcal{H}$  is Cohen–Macaulay and  $\dim \mathcal{H} = d + 1$ .*

*Proof.* Recall that  $*$  denotes the join of simplicial complexes and for a finite set  $S$  we denote by  $2^S$  the simplex with vertex set  $S$ . We set  $\deg x_i = \deg T = \deg z = 1$ . By definition  $\mathcal{H}$  is isomorphic to the quotient of the polynomial ring  $k[x_1, \dots, x_n, T, z]$  by a square-free monomial ideal. We denote by  $D_{\mathcal{H}}$  the simplicial complex on the vertex set  $\{1, 2, \dots, n, T, z\}$  that corresponds to the monomial ideal. The set of facets of  $D_{\mathcal{H}}$  is equal to the union

$$\{\{z, u\} : u \text{ facet of } D\} \cup \{\{T, 1, 2, \dots, q+1, w\} : w \text{ facet of } L\}.$$

As a consequence, we have the following decomposition

$$D_{\mathcal{H}} = E_1 \cup E_2$$

where  $E_1 = 2^{\{z\}} * D$ ,  $E_2 = 2^{\{T, 1, 2, \dots, q+1\}} * L$ .

Since  $D$  is Cohen–Macaulay over  $k$  with dimension  $d - 1$  we have that  $E_1$  is Cohen–Macaulay over  $k$  with dimension equal to  $d$ . Since  $L$  is Cohen–Macaulay over  $k$  with dimension  $d - q - 1$  we have that  $E_2$  is Cohen–Macaulay over  $k$  with dimension equal to  $d$ .

Moreover,  $E_1 \cap E_2 = 2^{\{1, 2, \dots, q+1\}} * L$  is also Cohen–Macaulay over  $k$  with dimension equal to  $d - 1$ . Hence using [9, Lemma 23.6] it follows that  $D_{\mathcal{H}}$  is Cohen–Macaulay over  $k$  with dimension  $d$ . Hence  $\mathcal{H}$  is a Cohen–Macaulay ring with dimension equal to  $d + 1$ .  $\square$

We denote by  $\mathcal{H}_a$  the ring  $\mathcal{H}$  but with  $\deg x_i = \deg T = 1$  and  $\deg z = q$ . Since the dimension and the Cohen–Macaulay property is independent of the grading, Proposition 4.16 implies that  $\mathcal{H}_a$  is Cohen–Macaulay and  $\dim \mathcal{H}_a = d + 1$ .

The element  $z - T^q \in \mathcal{H}_a$  is homogeneous and  $\mathcal{H}_a/(z - T^q) \cong R[T]/I_C$  as graded  $k$ -algebras. Hence  $\dim R[T]/I_C \geq d + 1 - 1 = d$ . By Lemma 4.7  $C$  is Artinian. Since  $C = R[T]/(I_C, f_1, \dots, f_d)$  it follows that  $\dim R[T]/I_C \leq d$ . As a consequence  $\dim R[T]/I_C = d$ . Since  $z - T^q$  is homogeneous,  $\mathcal{H}_a$  is Cohen–Macaulay and  $\dim \mathcal{H}_a/(z - T^q) = \dim \mathcal{H}_a - 1$ , it follows that

$z - T^q$  is  $\mathcal{H}_a$ -regular. Hence  $R[T]/I_C$  is Cohen–Macaulay. As a consequence,  $f_1, \dots, f_d$  is an  $R[T]/I_C$ -regular sequence. Hence  $\mathrm{HF}(R[T]/I_C) = \Gamma^d(\mathrm{HF}(C))$ .

Since  $I_G = (I, TI_L, T^{q+1} - x_\sigma)$  it is clear that  $I_C$  is a subset of the initial ideal of the  $I_G$  with respect to the reverse lexicographic order in  $R[T]$  with  $T > x_1 > \dots > x_n$ . Since by Lemma 4.7  $\mathrm{HF}(C)$  is equal to the Hilbert function of a general Artinian reduction of  $k[D_\sigma]$  and  $k[D_\sigma]$  is Gorenstein of dimension  $d$ , it follows that

$$\mathrm{HF}(R[T]/I_C) = \Gamma^d(\mathrm{HF}(C)) = \mathrm{HF}(k[D_\sigma]) = \mathrm{HF}(R[T]/I_G),$$

with the last equality by Lemma 4.9. As a consequence,  $I_C$  is equal to the initial ideal of the  $I_G$ . This finishes the proof of Lemma 4.10.

**4.2. Proof of Lemma 4.15.** By Lemma 4.6  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional and  $B = \bigoplus_{i=0}^{d-q-1} B_i$  with  $B_{d-q-1}$  1-dimensional. By Remark 8 for all  $i \geq 0$  we have

$$(3) \quad C_i = A_i \oplus (\bigoplus_{j=1}^q T^j B_{i-j})$$

Hence  $\dim C_i = \dim C_{d-i}$  for all  $i \in \mathbb{Z}$ . In particular  $C_i = 0$  for  $i \geq d+1$  and  $C_d$  is 1-dimensional. Let  $\omega \in R_1$  be a general linear element. By Lemma 4.14  $C$  has the WLP if and only if the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective.

We assume that  $A$  has the WLP and  $B$  satisfies property  $M_{q,p_1}$  and we will show that  $C$  has the WLP. For that we first show that the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective. Assume it is not. Then there exists  $0 \neq c \in C_{p_1}$  such that

$$(4) \quad (\omega + T)c = 0$$

in  $C_{p_1+1}$ . By Equation (3) there exist (unique)  $a_{p_1} \in A_{p_1}$  and, for  $1 \leq j \leq q$ ,  $b_{p_1-j} \in B_{p_1-j}$  such that

$$c = a_{p_1} + \sum_{j=1}^q b_{p_1-j} T^j$$

Since  $A$  is assumed to have the WLP, if  $a_{p_1} \neq 0$  we have  $\omega a_{p_1} \neq 0$  which implies  $(\omega + T)c \neq 0$ , which contradicts Equation (4). Hence we have  $a_{p_1} = 0$  in  $A$ . Equation (4) then implies that for  $j = 1, 2, \dots, q-1$

$$\omega b_{p_1-1} = 0, \quad b_{p_1-j} + \omega b_{p_1-(j+1)} = 0,$$

with all equations in  $B$ . Combining these equations we get  $\omega^q b_{p_1-q} = 0$  in  $B$ . Using the assumption that  $B$  satisfies property  $M_{q,p_1}$  it follows that  $b_{p_1-q} = 0$  in  $B$ , which using the above equations implies that  $b_{p_1-j} = 0$  for all  $1 \leq j \leq q$ , hence  $c = 0$ , a contradiction.

If  $d$  is odd, since  $p_1 = p_2$  and  $\dim C_{p_1} = \dim C_{p_1+1}$  we get that the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is also surjective, hence by what we said above  $C$  has the WLP. If  $d$  is even we need the following argument:

Assume  $c' \in C_{p_2+1}$  with

$$c' = a_{p_2+1} + \sum_{j=1}^q b_{p_2+1-j} T^j$$



where  $a_{p_2+1} \in A_{p_2+1}$  and  $b_{p_2+1-j} \in B_{p_2+1-j}$  for all  $1 \leq j \leq q$ . We will find

$$c = a_{p_2} + \sum_{j=1}^q e_{p_2-j} T^j \in C_{p_2}$$

where  $a_{p_2} \in A_{p_2}$  and  $e_{p_2-j} \in B_{p_2-j}$  for all  $1 \leq j \leq q$ . such that  $(\omega + T)c = c'$ . Hence we need to have (with the first equation in  $A$  and the remaining  $q$  equations in  $B$ )

$$\begin{aligned} a_{p_2+1} &= \omega a_{p_2} \\ b_{p_2+1-1} &= a_{p_2} + \omega e_{p_2-1} \\ b_{p_2+1-2} &= e_{p_2-1} + \omega e_{p_2-2} \\ &\vdots \\ b_{p_2+1-q} &= e_{p_2-q+1} + \omega e_{p_2-q} \end{aligned}$$

Since  $A$  is assumed to have the WLP, the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective, hence there exists  $a_{p_2} \in A_{p_2}$  such that  $a_{p_2+1} = \omega a_{p_2}$  in  $A$ . We fix such  $a_{p_2}$ . Given  $e_{p_2-q} \in B_{p_2-q}$ , the last  $q-1$  equations in  $B$  inductively determine (unique)  $e_{p_2-q+1}, \dots, e_{p_2-1}$  that satisfy them. What we need is to choose  $e_{p_2-q}$  in such a way that we have compatibility with the second equation

$$b_{p_2} = a_{p_2} + \omega e_{p_2-1}.$$

If we express  $e_{p_2-q+1}, \dots, e_{p_2-1}$  in terms of  $e_{p_2-q}$ , the compatibility equation becomes

$$\omega^q e_{p_2-q} = (-1)^q a_{p_2} + \sum_{i=0}^{q-1} (-1)^{q+1-i} \omega^i b_{p_2-i}$$

(equation in  $B$ ). Using the assumption that  $B$  satisfies property  $M_{q,p_1}$ , Remark 5 implies that there exists  $e_{p_2-q} \in B_{p_2-q}$  that satisfies the compatibility equation.

We now prove the converse. We assume that  $C$  has the WLP and we prove that  $A$  has the WLP and  $B$  satisfies the  $M_{q,p_1}$  property. Since  $C$  has the WLP, as we said above the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective.

Let  $a \in A_{p_2+1} \subset C_{p_2+1}$ . Then there exists  $c \in C_{p_2}$  such that  $a = (\omega + T)c$ . Write

$$c = a_{p_2} + \sum_{j=1}^q e_{p_2-j} T^j \in C_{p_2}$$

with equality in  $C$ , where  $a_{p_2} \in A_{p_2}$  and  $e_{p_2-j} \in B_{p_2-j}$  for all  $1 \leq j \leq q$ . It follows that  $a = \omega a_{p_2}$ , hence the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective. Using Remark 4 it follows that  $A$  has the WLP.

We will now prove that  $B$  has the property  $M_{q,p_1}$ . We assume it is not the case and we will get a contradiction. By the assumption there exists  $0 \neq b \in B_{p_1-q}$  such that  $\omega^q b = 0$  in  $B$ . We set  $c = \sum_{i=1}^q (-1)^{q-i} \omega^{q-i} b T^i \in C_{p_1}$ . Since the summand of  $c$  corresponding to  $i = q$  is  $b T^q$ , we get from

Equation (3) that  $c \neq 0$ . Using that  $T^{q+1} = 0$  in  $C$  we have

$$\begin{aligned} (\omega + T)c &= \sum_{i=1}^q (-1)^{q-i} \omega^{q-i+1} bT^i + \sum_{i=1}^q (-1)^{q-i} \omega^{q-i} bT^{i+1} \\ &= (-1)^{q-1} \omega^q bT + \sum_{i=2}^q (-1)^{q-i} \omega^{q-i+1} bT^i + \sum_{i=1}^{q-1} (-1)^{q-i} \omega^{q-i} bT^{i+1} \\ &= 0 \end{aligned}$$

which is a contradiction, since the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  was assumed to be injective. This contradiction finishes the proof of Lemma 4.15.

**4.3. Proof of Theorem 4.4.** We assume  $q > p_2$  and  $k[D_\sigma]$  has the WLP. The assumption that  $k[D_\sigma]$  has the WLP implies by Lemma 4.9 that  $G$  has the WLP. Moreover, by the same lemma we get that  $\text{HF}(G) = \text{HF}(C)$ . As a consequence, Lemma 4.7 implies that  $G_d$  is 1-dimensional and,  $G_j = 0$  for  $j > d$ . These properties, together with the Gorensteinness (by Lemma 4.9) and the WLP of  $G$  imply that for general  $\omega \in R_1$  the multiplication by  $\omega + T$  map  $G_{p_2} \rightarrow G_{p_2+1}$  is surjective.

Using that the map  $\psi$  in the proof of Lemma 4.9 is bijective, it follows that the natural map  $A \rightarrow G$ , with  $[a] \mapsto [a]$  for  $a \in R$ , is injective. Assume  $a \in A_{p_2+1} \subset G_{p_2+1}$ . Then there exists  $g \in G_{p_2}$  such that

$$(5) \quad (\omega + T)g = a$$

Using again that the map  $\psi$  in the proof of Lemma 4.9 is bijective, there exists  $e \in A_{p_2}$  and, for  $1 \leq j \leq q$ ,  $b_{p_2-j} \in B_{p_2-j}$  such that

$$g = e + \sum_{i=1}^q b_{p_2-i} T^i,$$

with equality in  $G$ . The assumption  $q > p_2$  implies that  $p_2 - q < 0$ , hence  $b_{p_2-q} = 0$  in  $B$ . As a consequence

$$a = (\omega + T)g = \omega e + \sum_{i=1}^{q-1} b_{p_2-i} T^i + \sum_{i=1}^{q-1} b_{p_2-i} T^{i+1}$$

(equality in  $G$ ), which imply that  $a = \omega e$  (equality in  $A \subset G$ ). It follows that the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective. Using Remark 4 it follows that  $A$  has the WLP. Hence, Lemma 4.11 implies that  $k[D]$  has the WLP. This finishes the proof of Theorem 4.4.

## 5. FURTHER RESULTS ON THE GENERAL QUESTION

In Corollary 4.5 we proved that if  $\sigma$  is a face of a Gorenstein\* simplicial complex  $D$  and  $2(\dim \sigma) > \dim D + 1$  then the Stanley–Reisner ring  $k[D]$  has the WLP if and only if  $k[D_\sigma]$  has the WLP. A natural question is whether the restriction on the dimension of  $\sigma$  is necessary. Proposition 5.4, which is motivated by Propositions 5.2 and 5.3, suggests some ideas that could, perhaps, prove useful in attacking this question. In addition, in Remark 12 we give a second proof of Corollary 4.5 based on the constructions of the present section.

We continue using the definitions and notation of Section 4. We set

$$\mathcal{L} = (I, f_1 I_L, f_2, \dots, f_{d+1}) \subset R, \quad \mathcal{P}_a = \mathcal{L} : x_\sigma \subset R,$$

$$\mathcal{P}_b = \mathcal{L} : f_1^{q+1} \subset R, \quad \mathcal{P}_c = (I_L, f_2, \dots, f_{d+1}) \subset R.$$

We also define the ideal  $I_Q = (I, T I_L) \subset R[T]$  and set  $h_1 = \text{HF}(k[D])$ ,  $h_2 = \text{HF}(R/I_L)$ ,  $\mathcal{K}_a = \mathcal{L} + (x_\sigma) \subset R$ ,  $\mathcal{K}_b = \mathcal{L} + (f_1^{q+1}) \subset R$ . There are two short exact sequences

$$(6) \quad 0 \rightarrow (R/\mathcal{P}_a)(-q-1) \rightarrow R/\mathcal{L} \rightarrow R/\mathcal{K}_a \rightarrow 0$$

and

$$(7) \quad 0 \rightarrow (R/\mathcal{P}_b)(-q-1) \rightarrow R/\mathcal{L} \rightarrow R/\mathcal{K}_b \rightarrow 0$$

where  $(-q-1)$  denotes twist by  $-q-1$ , the map  $R/\mathcal{P}_a \rightarrow R/\mathcal{L}$  is multiplication by  $x_\sigma$  and the map  $R/\mathcal{P}_b \rightarrow R/\mathcal{L}$  is multiplication by  $f_1^{q+1}$ .

We also have

$$(8) \quad R/\mathcal{K}_a = k[D_\sigma]/(T - f_1, f_2, \dots, f_{d+1}),$$

and

$$(9) \quad R/\mathcal{K}_b = (R[T]/I_C)/(T - f_1, f_2, \dots, f_{d+1}).$$

The following proposition gives the main properties of  $R[T]/I_Q$ .

**Proposition 5.1.** *i) We have  $\dim R[T]/I_Q = d+1$  and  $\text{depth } R[T]/I_Q = d$ .  
ii) For all  $m \geq 0$  we have*

$$\text{HF}(m, R[T]/I_Q) = \text{HF}(m, k[D]) + \sum_{j=1}^m \text{HF}(m-j, R/I_L)$$

*iii) If  $t \leq d$  then the sequence  $T - f_1, f_2, \dots, f_t$  is  $R[T]/I_Q$ -regular.*

*Proof.* i) Denote by  $D_Q$  the simplicial complex that corresponds to the square-free monomial ideal  $I_Q$ . We set  $P_1 = (I, T) \subset R[T]$ ,  $P_2 = (I_L) \subset R[T]$ . Using that  $I \subset I_L$  and that  $T$  is a new variable we have  $I_Q = P_1 \cap P_2$ . Hence

$$D_Q = D \cup (2^{\{T, 1, \dots, q+1\}} * L).$$

As a consequence, since  $\dim D = d-1$  and  $\dim L = d-q-2$  we have  $\dim D_Q = d$ , hence  $\dim R[T]/I_Q = d+1$ . Since  $D_Q$  is not pure (in the sense that it contains facets of different dimensions), by [2, Corollary 5.1.5]  $R[T]/I_Q$  is not Cohen–Macaulay, hence  $\text{depth } R[T]/I_Q \leq d$ .

We have that  $R[T]/P_1$  is Cohen–Macaulay of dimension  $d$ , hence has depth  $d$ ,  $R[T]/P_2$  is Cohen–Macaulay of dimension  $d+1$ , hence has depth  $d+1$  and  $R[T]/(P_1 + P_2)$  is Cohen–Macaulay of dimension  $d$ , hence has depth  $d$ . Using Equation (1) and the additivity of the Ext functor on the second variable ([20, Proposition 3.3.4]) we get that  $\text{depth}(R[T]/P_1 \oplus R[T]/P_2) = d$ .

Since  $I_Q = P_1 \cap P_2$ , there is, by [2, Proof of Theorem 5.1.13], a short exact sequence of  $R$ -modules

$$(10) \quad 0 \rightarrow R[T]/I_Q \rightarrow R[T]/P_1 \oplus R[T]/P_2 \rightarrow R[T]/(P_1 + P_2) \rightarrow 0$$

As a consequence, using [2, Proposition 1.2.9] we get

$$\text{depth } R[T]/I_Q \geq \text{depth}(R[T]/P_1 \oplus R[T]/P_2) = d.$$

Since we proved above that  $\text{depth } R[T]/I_Q \leq d$ , we get  $\text{depth } R[T]/I_Q = d$ .

ii) Assume  $m \geq 0$ . Using the short exact sequence (10) we have

$$\mathrm{HF}(m, R[T]/I_Q) = \mathrm{HF}(m, R[T]/P_1) + \mathrm{HF}(m, R[T]/P_2) - \mathrm{HF}(m, R[T]/(P_1 + P_2))$$

Since  $\mathrm{HF}(m, R[T]/P_1) = h_1(m)$ ,  $\mathrm{HF}(m, R[T]/P_2) = (\Gamma^1(h_2))(m) = \sum_{j=0}^m h_2(j)$  and  $\mathrm{HF}(m, R[T]/(P_1 + P_2)) = h_2(m)$  the result follows.

iii) Since  $\mathrm{depth} R[T]/I_Q = d$ ,  $t \leq d$ , and the ideal of  $R[T]$  generated by  $T - f_1, f_2, \dots, f_t$  is equal to the ideal of  $R[T]$  generated by  $t$  general linear elements, Lemma 3.3 implies that the sequence  $T - f_1, f_2, \dots, f_t$  is  $R[T]/I_Q$ -regular.  $\square$

The following two propositions motivate Proposition 5.4.

**Proposition 5.2.** *For all  $m \geq 0$  we have*

$$(11) \quad \mathrm{HF}(m, R/\mathcal{P}_b) \leq \mathrm{HF}(m, R/\mathcal{P}_a).$$

*Proof.* Since  $f_1, \dots, f_{d+1}$  are general linear elements of  $R$  it follows that the ideal of  $R[T]$  generated by  $T - f_1, f_2, \dots, f_{d+1}$  is equal to the ideal of  $R[T]$  generated by  $d+1$  general linear elements. Since by the proof of Lemma 4.12  $I_C$  is an initial ideal of  $J_{st}$  (with respect to a suitable monomial order) by [4, Theorem 1.1] we have

$$\mathrm{HF}(m, k[D_\sigma]/(T - f_1, f_2, \dots, f_{d+1})) \leq \mathrm{HF}(m, R[T]/(I_C, T - f_1, f_2, \dots, f_{d+1}))$$

for all  $m \geq 0$ . Hence, Equations (8) and (9) imply that

$$\mathrm{HF}(m, R/\mathcal{K}_a) \leq \mathrm{HF}(m, R/\mathcal{K}_b)$$

for all  $m \geq 0$ . As a consequence, the short exact sequences (6) and (7) imply Inequality (11).  $\square$

**Proposition 5.3.** *Assume  $1 \leq t \leq d$ . Then we have the following equality of ideals of  $R$*

$$(I, f_1 I_L, f_2, \dots, f_t) : x_\sigma = (I, f_1 I_L, f_2, \dots, f_t) : f_1^{q+1} = (I_L, f_2, \dots, f_t).$$

*Proof.* Recall  $h_2 = \mathrm{HF}(R/I_L)$ . We first prove the equality for  $t = 1$ . For simplicity we set  $h_3 = \mathrm{HF}(R[T]/I_Q)$ ,  $h_4 = \mathrm{HF}(k[D_\sigma])$ ,

$$J_1 = (I, f_1 I_L) : x_\sigma, \quad J_2 = (I, f_1 I_L) : f_1^{q+1}.$$

Since it is clear that, for  $i = 1, 2$ , we have  $I_L \subset J_i$ , to prove  $J_1 = J_2 = I_L$  it is enough to prove that

$$\mathrm{HF}(R/J_1) = \mathrm{HF}(R/J_2) = h_2.$$

Consider the short exact sequence

$$(12) \quad 0 \rightarrow (R/J_1)(-q-1) \rightarrow R[T]/(I_Q, T - f_1) \rightarrow R[T]/(I_Q, T - f_1, x_\sigma) \rightarrow 0$$

where  $(-q-1)$  means twist by  $-q-1$  and the first map is multiplication by  $x_\sigma$ . We have  $R[T]/(I_Q, x_\sigma) = k[D_\sigma]$ . Moreover, the ideal of  $R[T]$  generated by  $T - f_1$  is equal to the ideal generated by a general linear element of  $R[T]$ . Since  $k[D_\sigma]$  is Gorenstein of dimension  $d$  it follows from Lemma 3.3 that  $T - f_1$  is  $k[D_\sigma]$ -regular, hence

$$\mathrm{HF}(R[T]/(I_Q, T - f_1, x_\sigma)) = \Delta^1(h_4)$$

Since by Proposition 5.1  $\text{depth } R[T]/I_Q = d$ , it follows from Lemma 3.3 that  $T - f_1$  is  $R[T]/I_Q$ -regular, hence

$$\text{HF}(R[T]/(I_Q, T - f_1)) = \Delta^1(h_3).$$

As a consequence, the short exact sequence (12) implies that

$$(13) \quad \text{HF}((R/J_1)(-q-1)) = \Delta^1(h_3 - h_4)$$

Combining the computation of  $h_3$  in Proposition 5.1 and the computation of  $h_4$  in Lemma 4.6. we get that, for all  $m \geq 0$ ,

$$(h_3 - h_4)(m) = \sum_{j=q+1}^m h_2(m-j) = \sum_{j=0}^{m-q-1} h_2(j)$$

Hence for all  $m \geq 0$

$$(\Delta^1(h_3 - h_4))(m) = h_2(m - q - 1).$$

As a consequence, Equation (13) implies that  $\text{HF}(m, R/J_1) = h_2(m)$  for all  $m \geq 0$ .

Consider now the short exact sequence

$$(14) \quad 0 \rightarrow (R/J_2)(-q-1) \rightarrow R[T]/(I_Q, T - f_1) \rightarrow R[T]/(I_Q, T - f_1, T^{q+1}) \rightarrow 0$$

We have  $R[T]/(I_Q, T^{q+1}) = R[T]/I_C$ , which by Lemma 4.10 is Cohen–Macaulay with same Hilbert function as  $k[D_\sigma]$ . Since the ideal of  $R[T]$  generated by  $T - f_1$  is equal to the ideal generated by a general linear element of  $R[T]$ , we get

$$\text{HF}(R[T]/(I_Q, T - f_1, f_1^{q+1})) = \Delta^1(h_4).$$

As a consequence, using the previously done computations of  $\text{HF}(R[T]/(I_Q, T - f_1))$  and  $\Delta^1(h_3 - h_4)$ , the short exact sequence (14) implies that  $\text{HF}(m, R/J_2) = h_2(m)$  for all  $m \geq 0$ . This finishes the proof of the double equality  $J_1 = J_2 = I_L$ .

We now assume  $2 \leq t \leq d$ . We set

$$J_4 = (I, f_1 I_L, f_2, \dots, f_t) : x_\sigma, \quad J_5 = (I, f_1 I_L, f_2, \dots, f_t) : f_1^{q+1},$$

$$J_6 = (I_L, f_2, \dots, f_t).$$

Clearly, for  $i = 4, 5$ , we have that  $J_6 \subset J_i$ . Hence, to prove  $J_4 = J_5 = J_6$  it is enough to prove that  $\text{HF}(R/J_4) = \text{HF}(R/J_5) = \text{HF}(R/J_6)$ . To prove this equality, it is enough to prove that

$$\text{HF}(R/J_4) = \Delta^{t-1} \text{HF}(R/J_1), \quad \text{HF}(R/J_5) = \Delta^{t-1} \text{HF}(R/J_2),$$

and

$$\text{HF}(R/J_6) = \Delta^{t-1} \text{HF}(R/I_L),$$

since, as we proved above,  $\text{HF}(R/J_1) = \text{HF}(R/J_2) = \text{HF}(R/I_L)$ .

By Lemma 4.6  $R/I_L$  is Gorenstein with  $\dim R/I_L = d$ . Since  $t \leq d$  and  $f_2, \dots, f_t$  are general linear elements over an infinite field, it follows by Lemma 3.3 that they are a regular  $R/I_L$  sequence, hence  $\text{HF}(R/J_6) = \Delta^{t-1} \text{HF}(R/I_L)$ .

For  $J_4$  we have the short exact sequence

$$0 \rightarrow R/J_4(-q-1) \rightarrow R/(I, f_1 I_L, f_2, \dots, f_t) \rightarrow R/(I, f_1 I_L, f_2, \dots, f_t, x_\sigma) \rightarrow 0$$

while for  $J_5$  we have the short exact sequence

$$0 \rightarrow R/J_4(-q-1) \rightarrow R/(I, f_1 I_L, f_2, \dots, f_t) \rightarrow R/(I, f_1 I_L, f_2, \dots, f_t, f_1^{q+1}) \rightarrow 0$$

Hence, it is enough to prove the following three equalities

$$\mathrm{HF}(R/(I, f_1 I_L, f_2, \dots, f_t)) = \Delta^{t-1} \mathrm{HF}(R/(I, f_1 I_L)),$$

$$\mathrm{HF}(R/(I, f_1 I_L, f_2, \dots, f_t, x_\sigma)) = \Delta^{t-1} \mathrm{HF}(R/(I, f_1 I_L, x_\sigma)),$$

and

$$\mathrm{HF}(R/(I, f_1 I_L, f_2, \dots, f_t, f_1^{q+1})) = \Delta^{t-1} \mathrm{HF}(R/(I, f_1 I_L, f_1^{q+1})).$$

Each of the three equalities follows using Lemma 3.3, taking into account that  $t \leq d$ , the ideal  $(T - f_1, f_2, f_3, \dots, f_t)$  of  $R[T]$  is equal to the ideal of  $R[T]$  generated by  $t$  general linear elements, and that  $\mathrm{depth} R[T]/I_Q = d$  (by Proposition 5.1),  $\mathrm{depth} k[D_\sigma] = d$  (since  $k[D_\sigma]$  is Gorenstein of dimension  $d$ ) and  $\mathrm{depth} R[T]/I_C = d$  (since by Lemma 4.10  $R[T]/I_C$  is Cohen–Macaulay of dimension  $d$ ).  $\square$

**Remark 9.** As we mention below in Remark 10, for  $t = d + 1$ , which is the critical value for the WLP properties, the triple equality of ideals in the statement of Proposition 5.3 does not hold any more.

For a homogeneous ideal  $J \subset R$  and  $t \geq 0$  we denote by  $J_{\leq t}$  the ideal of  $R$  generated by all homogeneous elements of  $J$  that have degree  $\leq t$ . The following proposition is motivated by Propositions 5.2 and 5.3.

**Proposition 5.4.** *i) Assume  $k[D_\sigma]$  has the WLP. If  $\mathrm{HF}(R/\mathcal{P}_a) = \mathrm{HF}(R/\mathcal{P}_b)$  then  $k[D]$  has the WLP.*

*ii) Assume  $k[D_\sigma]$  has the WLP and condition  $M_{q,p_1}$  holds for  $k[L]$ . Then  $k[D]$  has the WLP if and only if  $\mathrm{HF}(R/\mathcal{P}_a) = \mathrm{HF}(R/\mathcal{P}_b)$ .*

*iii) Assume  $k[D_\sigma]$  has the WLP and  $d$  is odd. Assume the following equality of ideals of  $R$  holds:*

$$(\mathcal{P}_b)_{\leq p_1 - q} = (\mathcal{P}_c)_{\leq p_1 - q}.$$

*Then  $k[D]$  has the WLP.*

*Proof.* We first prove i). The short exact sequences (6) and (7) imply that  $\mathrm{HF}(R/\mathcal{K}_a) = \mathrm{HF}(R/\mathcal{K}_b)$  if and only if  $\mathrm{HF}(R/\mathcal{P}_a) = \mathrm{HF}(R/\mathcal{P}_b)$ . By Equation (8)

$$R/\mathcal{K}_a = k[D_\sigma]/(T - f_1, f_2, \dots, f_{d+1}).$$

Since the ideal  $(T - f_1, \dots, f_{d+1})$  is equal to the ideal of  $R[T]$  generated by  $d + 1$  general linear elements, the assumption that  $k[D_\sigma]$  has the WLP and the fact (see Lemma 4.7) that  $\mathrm{HF}(C)$  is equal to the HF of a general Artinian reduction of  $k[D_\sigma]$  imply that

$$\mathrm{HF}(R/\mathcal{K}_a) = \Delta^+(\mathrm{HF}(C)).$$

Hence from the assumption  $\mathrm{HF}(R/\mathcal{P}_a) = \mathrm{HF}(R/\mathcal{P}_b)$  it follows that

$$\mathrm{HF}(R/\mathcal{K}_b) = \mathrm{HF}(R/\mathcal{K}_a) = \Delta^+(\mathrm{HF}(C))$$

hence  $C$  has the WLP. Lemma 4.15 implies that  $A$  has the WLP. Hence, by Lemma 4.11  $k[D]$  has the WLP.

ii) We assume that  $k[D_\sigma]$  has the WLP, condition  $M_{q,p_1}$  holds for  $k[L]$  and  $\text{HF}(R/\mathcal{P}_a) = \text{HF}(R/\mathcal{P}_b)$ . From Part i) of the present proposition we get that  $k[D]$  has the WLP.

We now assume that both  $k[D_\sigma]$  and  $k[D]$  have the WLP and condition  $M_{q,p_1}$  holds for  $k[L]$  and we will prove that  $\text{HF}(R/\mathcal{P}_a) = \text{HF}(R/\mathcal{P}_b)$ . Combining Lemmas 4.11 and 4.15 we get that  $C$  has the WLP. Hence both  $k[D_\sigma]$  and  $C$  have the WLP. Since by Lemma 4.7  $\text{HF}(C)$  is equal to the HF of a general Artinian reduction of  $k[D_\sigma]$ , we get that  $\text{HF}(R/\mathcal{K}_b) = \text{HF}(R/\mathcal{K}_a)$  which, using the two short exact sequences (6) and (7), implies  $\text{HF}(R/\mathcal{P}_a) = \text{HF}(R/\mathcal{P}_b)$ .

iii) Since  $d$  is odd we have  $p_1 = p_2 = (d-1)/2$ . Since always  $\mathcal{P}_c \subset \mathcal{P}_a$ , we get that for all  $m \geq 0$

$$\text{HF}(m, R/\mathcal{P}_a) \leq \text{HF}(m, R/\mathcal{P}_c).$$

Moreover, by Proposition 5.2 we have that  $\text{HF}(m, R/\mathcal{P}_b) \leq \text{HF}(m, R/\mathcal{P}_a)$  for all  $m \geq 0$ . Using that  $p_1 = p_2$  and the assumption  $(\mathcal{P}_b)_{\leq p_1-q} = (\mathcal{P}_c)_{\leq p_1-q}$  we get

$$\text{HF}(i, R/\mathcal{P}_a) = \text{HF}(i, R/\mathcal{P}_b)$$

for all  $i \leq p_2 - q$ . As a consequence, the two short exact sequences (6) and (7) imply that

$$(15) \quad \text{HF}(i, R/\mathcal{K}_a) = \text{HF}(i, R/\mathcal{K}_b)$$

for all  $i \leq p_2 - q + q + 1 = p_2 + 1$ . The assumption that  $k[D_\sigma]$  has the WLP gives  $\text{HF}(p_2 + 1, R/\mathcal{K}_a) = 0$ . Hence, Equation (15) implies  $\text{HF}(p_2 + 1, R/\mathcal{K}_b) = 0$ . As a consequence, Equation (15) implies  $\text{HF}(R/\mathcal{K}_a) = \text{HF}(R/\mathcal{K}_b)$ . Hence, Corollary 4.8 implies that  $C$  has the WLP, which by Lemma 4.15 implies that  $A$  has the WLP which by Lemma 4.11 implies that  $k[D]$  has the WLP.  $\square$

**Remark 10.** It is clear that we always have  $\mathcal{P}_c \subset \mathcal{P}_a \cap \mathcal{P}_b$ . Macaulay2 [7] computations for some 1-faces of the boundary complex of the cyclic polytope with 10 vertices and dimension 6 give examples such that  $\mathcal{P}_a, \mathcal{P}_b$  have the same HF but are not equal as ideals of  $R$  and, moreover, the ideal  $\mathcal{P}_c$  is a proper subset of  $\mathcal{P}_a \cap \mathcal{P}_b$ .

**Remark 11.** The present remark is related to Part iii) of Proposition 5.4. Macaulay2 computations suggests that independently of whether  $d$  is even or odd the inequality  $(\mathcal{P}_b)_{\leq p_1-q} = (\mathcal{P}_c)_{\leq p_1-q}$  perhaps holds. Can it be proven theoretically? However, when  $d$  is even it seems that even if we assume that  $k[D_\sigma]$  has the WLP and  $(\mathcal{P}_b)_{\leq p_1-q} = (\mathcal{P}_c)_{\leq p_1-q}$  it is not clear how to conclude that  $k[D]$  has the WLP.

**Remark 12.** Assume  $q > p_2$ . We give a second proof of Corollary 4.5 that does not use the ring  $R[T]/I_G$ . The two short exact sequences (6) and (7) imply that  $\text{HF}(i, R/\mathcal{K}_a) = \text{HF}(i, R/\mathcal{K}_b)$  for all  $0 \leq i \leq q$ . By Equation (8)  $R/\mathcal{K}_a$  is isomorphic to a general linear section of a general Artinian reduction of  $k[D_\sigma]$ , while by Equation (9)  $R/\mathcal{K}_b$  is isomorphic to a general linear section of  $C$ .

Assume first that  $k[D_\sigma]$  has the WLP, then  $\text{HF}(p_2 + 1, R/\mathcal{K}_a) = 0$ . Since  $q > p_2$  we get that  $\text{HF}(p_2 + 1, R/\mathcal{K}_b) = 0$ . As a consequence,  $\text{HF}(R/\mathcal{K}_a) =$

$\text{HF}(R/\mathcal{K}_b)$ . Hence, Corollary 4.8 implies that  $C$  has the WLP, which by Lemma 4.15 implies that  $A$  has the WLP which by Lemma 4.11 implies that  $k[D]$  has the WLP.

Conversely assume that  $k[D]$  has the WLP. By Lemma 4.11  $A$  has the WLP. Since  $q > p_2$  and  $p_2 \geq p_1$  by Lemma 4.1 property  $M_{q,p_1}$  holds for  $B$ . Hence Lemma 4.15 implies that  $C$  has the WLP. Then  $\text{HF}(p_2 + 1, R/\mathcal{K}_b) = 0$ . Since  $q > p_2$  we get that  $\text{HF}(p_2 + 1, R/\mathcal{K}_a) = 0$ . As a consequence,  $\text{HF}(R/\mathcal{K}_a) = \text{HF}(R/\mathcal{K}_b)$ . Hence, Corollary 4.8 implies that  $k[D_\sigma]$  has the WLP. This finishes the second proof of Corollary 4.5.

### 5.1. An interesting example not coming from simplicial complexes.

The following example is related to Remark 15 below and is taken from [19, Example 2.6]. Assume  $k$  is an infinite field of characteristic 0 and  $R = k[x_1, \dots, x_4]$  with the degrees of all variables equal to 1. Consider the homogeneous ideal

$$I = (x_1^3, x_2^3, x_3^3, x_4^3, (x_1 + x_2 + x_3 + x_4)^3) \subset R$$

The ideal  $I$  is an Artinian (hence Cohen–Macaulay) codimension 4 almost complete intersection without the WLP. We set  $I_L = (I : x_1 x_2) \subset R$ . Macaulay2 computations suggest that the Artinian ideal  $(I, T^2, TI_L) \subset R[T]$  does not have the WLP, while the Artinian ideal  $(I, T^2 - x_1 x_2, TI_L) \subset R[T]$  does have the WLP.

**5.2. Final Remarks.** In the following remarks we keep assuming that  $D$  is a Gorenstein\* simplicial complex and  $k$  is an infinite field.

**Remark 13.** Suppose

$$D_0 = D, D_1, \dots, D_m$$

is a finite sequence of simplicial complexes such that, for all  $0 \leq i \leq m - 1$ , the complex  $D_{i+1}$  is obtained from  $D_i$  by a stellar subdivision with respect to a face  $\sigma_i$  of  $D_i$  with  $2(\dim \sigma_i) > \dim D + 1$ . Then, by Corollary 4.5 the Stanley–Reisner ring  $k[D]$  has the WLP if and only if  $k[D_m]$  has the WLP. Is it possible to prove that starting from  $D$  there exists a sequence of stellar subdivisions as above with  $k[D_m]$  WLP? Then it would follow that  $k[D]$  has the WLP. Compare also [11, Conjecture 4.12].

**Remark 14.** Recall  $I_C = (I, T^{q+1}, TI_L)$ . Assume that  $k[D_\sigma]$  has the WLP. Is it possible to prove that  $R[T]/I_C$  has the WLP? If so, combining Lemmas 4.11 and 4.15 it would then follow that  $k[D]$  has the WLP.

**Remark 15.** Recall  $I_G = (I, T^{q+1} - x_\sigma, TI_L)$ ,  $G = R[T]/(I_G + (f_1, \dots, f_d))$ . Assume  $k[D_\sigma]$  has the WLP. Then by Lemma 4.9  $R[T]/I_G$  and  $G$  have the WLP. But it is not clear how to use that to prove that  $k[D]$  has the WLP. Compare also the example in Subsection 5.1 which does not come from simplicial complexes.

In general  $G$  WLP only implies  $\text{HF}(G/(f_{d+1} + T)) = \Delta^+(\text{HF}(G))$ . However, some Macaulay2 computational evidence suggests that perhaps in the Gorenstein\* setting it holds that

$$(16) \quad \text{HF}(G/(f_{d+1})) = \Delta^+(\text{HF}(G)).$$

Is it possible to theoretically prove Equation (16)? Assume Equation (16) holds. Then the multiplication by  $f_{d+1}$  map  $G_{p_1} \rightarrow G_{p_1+1}$  is injective. Since



by the proof of Lemma 4.9  $A_i \subset G_i$  for all  $i$ , we get that the multiplication by  $f_{d+1}$  map  $A_{p_1} \rightarrow A_{p_1+1}$  is injective. By Remark 4  $A$  has the WLP, hence by Lemma 4.11  $k[D]$  has the WLP.

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